Lecture 26

11.6 - Absolute Convergence and the Root and Ratio Tests

Absolute Convergence

A series $\sum a_n$ is called absolutely convergent if

Ex: Are the series absolutely convergent?

ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n) + (-1)^{n-1}}{n^2}$ ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Def: A series which is convergent, but not absolutely convergent is called conditionally convergent.

Theorem: If $\sum a_n$ is absolutely convergent, then it is convergent.

proof:

Ex: Do the series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{17}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^{3}}$

Ex: When is the series $\sum_{n=1}^{\infty} ar^n$ absolutely convergent?

The Ratio Test: Consider the series $\sum_{n=1}^{\infty} a_n$, and let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

This test is useful when we have factorials and powers of constants hanging around.

Ex: Test the following series for convergence:

(a) $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ (b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1.1)^n}{n^4}$ (c) $\sum_{n=6}^{\infty} \frac{1}{n^3}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n}$

using the ratio test.

The Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$ and let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$,

Ex: Use the root test on the series:

ⓐ $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ ⓑ $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ ⓒ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

Ex: In section 11.10, we will learn how to show that

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

For what values of x is this a valid equation? (I.e., the series is convergent.) Does the convergence type (absolute/conditional) depend on x?

Rearrangements

The convergence of a series which is conditionally convergent is quite delicate... The last example shows

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

but, if we arrange the terms as:

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \cdots$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots\right]$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{2} \ln 2$$

Interesting Facts

- If $\sum a_n$ is abs. conv. & $\sum a_n = S$, then any rearrangement of terms still sums to S.
- If $\sum a_n$ is cond. conv., then for any real number r, we can find a rearrangement so that the sum is r.